6.9 Trigonometric Identities

Practice Tasks

I. Trigonometric Identities

Let's ponder the following question: "For what values of θ is the equation $\sin\theta = 1$ true?" The easiest answer, of course, would be $\theta = \frac{\pi}{2}$. However, that is not the only answer. If we substitute in $\theta = \frac{5\pi}{2}$ or $-\frac{3\pi}{2}$, the equation would still be true. In fact, there are an infinite number of solutions, as long as you start with $\frac{\pi}{2}$ and add or subtract 2π as many times as you like. Even though there are an infinite number of solution, this does not mean we can plug in *any* value for θ. For instance, if we set θ equal to $\frac{\pi}{3}$, we know that $\sin\frac{\pi}{3}$ does not equal 1.

Unlike $\sin\theta = 1$, there are in fact trigonometric equations that hold true for any value of θ. These trigonometric equations are called <u>trigonometric identities</u>. A good example of a trigonometric identity is the equation $\sin^2\theta + \cos^2\theta = 1$. You can test it out for yourself: substitute in any value for θ, and you will always end up with $1 = 1$.

Here are some of the most common trigonometric identities that you will see:

Pythagorean Identities:

$\sin^2\theta + \cos^2\theta = 1$ $\qquad$ $\tan^2\theta + 1 = \sec^2\theta$ $\qquad$ $\cot^2\theta + 1 = \csc^2\theta$

Addition and Subtraction Formulas:

$\sin(s+t) = \sin s\cos t + \cos s\sin t$ $\qquad$ $\sin(s-t) = \sin s\cos t - \cos s\sin t$

$\cos(s+t) = \cos s\cos t - \sin s\sin t$ $\qquad$ $\cos(s-t) = \cos s\cos t + \sin s\sin t$

$$\tan(s+t) = \frac{\tan s + \tan t}{1 - \tan s\tan t} \qquad \tan(s-t) = \frac{\tan s - \tan t}{1 + \tan s\tan t}$$

In the next section, we will use the common trigonometric identities, as well as the definition of trigonometric functions, in order to prove new identities.

II. Proving Trigonometric Identities

When we use the phrase "prove the identity," it means that we want to show that both sides of the equation are equal no matter what number you substitute into the variable. However, it would be impossible to substitute in *every* number into the variable. Luckily, there is a pretty straightforward way to prove identities. Start with one side of the equation and rearrange it using known identities (listed on the first page), and the definition of trigonometric functions in order to end up with the other side of the equation. Let's start off with an easier identity:

1. Prove the identity $\sin x = \tan x \cos x$

step 1: rewrite $\tan x$ in terms of sine and cosine

step 2: simplify the expression on the right side of the equation by canceling out the cosines

Not so bad, right? The problems will get more complex, but there are always a few different techniques to try.

<u>Guidelines for Proving Trigonometric Identities</u>:

1. Start with one side. Pick one side of the equation and write it down (the more complex side is often the easiest to start with). Your goal is to transform that side into the other side.

2. Use known identities. Use algebra and your known identities (such as the Pythagorean identities) to change the side you started with. If there are fractions, find a common denominator and combine them into one fraction. You may also need to use factoring and canceling out terms in order to simplify the expression.

3. Convert to sines and cosines. If you are stuck, try converting all functions in terms of sines and cosines. For example, if you have the expression $\tan x \csc x$, change it to $\left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\sin x}\right)$. Notice that the $\sin x$ terms will cancel out.

Let's try a couple more proof problems:
Prove the following identities

2. $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

3. $\frac{1+\tan x}{1-\tan x} = \tan(\frac{\pi}{4} + x)$

There are actually a few more common trigonometric identities that we will use, in addition to the Pythagorean Identities and Addition/Subtraction Identities listed in the box on the first page. However, rather than just give you the identities, we are now able to *prove* them using the Guidelines for Proving Trigonometric Identities.

Prove the following identities using the Subtraction Identities:

4. $\sin(\frac{\pi}{2} - u) = \cos u$

5. $\cos(\frac{\pi}{2} - u) = \sin u$

6. $\tan(\frac{\pi}{2} - u) = \cot u$

These three identities are called cofunction identities. The six cofunction identities are listed below:

Cofunction Identities:

$\sin(\frac{\pi}{2} - u) = \cos u$ $\cos(\frac{\pi}{2} - u) = \sin u$ $\tan(\frac{\pi}{2} - u) = \cot u$

$\csc(\frac{\pi}{2} - u) = \sec u$ $\sec(\frac{\pi}{2} - u) = \csc u$ $\cot(\frac{\pi}{2} - u) = \tan u$

The final identities we will look at are called the <u>double-angle formulas</u>. Let's prove them!
(Hint: start with the left side of the equation and rewrite $\sin 2x$ and $\cos 2x$ as $\sin(x + x)$ and $\cos(x + x)$ respectively.

7. $\sin 2x = 2 \sin x \cos x$

8. $\cos 2x = \cos^2 x - \sin^2 x$

There are actually three different ways to write the double-angle formula for cosine. They are all written below.

Double-Angle Formulas:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

III. Practice

The following trigonometric proof problems will require you to use the many trigonometric identities we have learned in this section. Sometimes, you will have to use more than one identity to solve the proof. If you get stuck, don't get discouraged! These problems can be very difficult…try starting over and using a different technique.

9. $(\sin x + \cos x)^2 = 1 + \sin 2x$

10. $\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$

11. $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2 \sec x \tan x$

12. $\frac{1+\sin 2x}{\sin 2x} = 1 + \frac{1}{2} \sec x \csc x$